

Operator Algebra of the  $SL(2)$  conformal field theoriesOleg Andreev<sup>\*†</sup>L.D.Landau Institute for Theoretical Physics,  
Kosygina 2, 117940 Moscow, Russia**Abstract**

Structure constants of Operator Algebras for the  $SL(2)$  degenerate conformal field theories are calculated.

Since the seminal work of Belavin, Polyakov and Zamolodchikov [1], there has been much progress in understanding two-dimensional conformal field theories. It is essential to compute the structure constants of Operator Algebra of such theories. In fact, it was done only for relatively few theories. The most famous examples are the diagonal minimal models and  $SU(2)$  WZW theory [2, 3]. There are also works on the structure constants of non diagonal theories (see e.g.[4] and refs. therein).

Degenerate conformal theories attract special interest since they are known to describe critical fluctuations in statistical models. From a mathematical point of view they correspond to reducible (with singular vectors) representations of chiral algebras. An irreducible representations are obtained by setting the singular vectors to zero that leads to differential equations for correlation functions [1].

In this paper, I shall compute the structure constants of Operator Algebra for the  $SL(2)$  degenerate conformal field theories. Such theories contain the reducible representations of the chiral algebra  $\hat{sl}_2$  with the highest weights listed by Kac-Kazhdan (see (5) below). The well-known integrable representations are a special case, namely  $j_{1,m}^+ k \in \mathbb{N}$ . These theories are of great interest because of their relevance in 2d quantum gravity coupled to conformal minimal matter as well as their connection via a quantum hamiltonian reduction with Dotsenko-Fateev (DF) models.

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The theories have  $\hat{sl}_2 \times \hat{sl}_2$  algebra as the symmetry algebra. The chiral currents of  $\hat{sl}_2$  form the following OP algebra

$$J^\alpha(z_1)J^\beta(z_2) = \frac{k}{2}q^{\alpha\beta}\frac{1}{(z_1 - z_2)^2} + \frac{f_\gamma^{\alpha\beta}}{(z_1 - z_2)}J^\gamma(z_2) + O(1) \quad , \quad (1)$$

where  $k$  is the level,  $q^{00} = 1$ ,  $q^{+-} = q^{-+} = 2$ ,  $f_+^{0+} = f_-^{-0} = 1$ ,  $f_0^{+-} = 2$ ;  $\alpha, \beta = 0, +, -$ . The same OP expansions, of course, are valid for antiholomorphic currents.

The stress-energy tensor of the models has two independent components which can be chosen in the Sugawara form

$$\begin{aligned} T(z) &= \frac{1}{k+2}q_{\alpha\beta} : J^\alpha(z)J^\beta(z) : \quad , \\ \bar{T}(\bar{z}) &= \frac{1}{k+2}q_{\alpha\beta} : \bar{J}^\alpha(\bar{z})\bar{J}^\beta(\bar{z}) : \quad . \end{aligned} \quad (2)$$

The primary fields (basic conformal operators) are defined as

$$\begin{aligned} J^\alpha(z_1)\Phi^{j\bar{j}}(z_2, \bar{z}_2) &= \frac{S_j^\alpha}{z_1 - z_2}\Phi^{j\bar{j}}(z_2, \bar{z}_2) + O(1) \quad , \\ \bar{J}^\alpha(\bar{z}_1)\Phi^{j\bar{j}}(z_2, \bar{z}_2) &= \frac{\bar{S}_{\bar{j}}^\alpha}{\bar{z}_1 - \bar{z}_2}\Phi^{j\bar{j}}(z_2, \bar{z}_2) + O(1) \quad . \end{aligned} \quad (3)$$

Here  $S_j^\alpha(\bar{S}_{\bar{j}}^\alpha)$  is the "left"("right") representation of  $sl_2$ . The conformal dimensions of  $\Phi^{j\bar{j}}$  follow from the OP expansions with  $T(z)$ ,  $\bar{T}(\bar{z})$ . They are given by  $\Delta_j = \frac{j(j+1)}{k+2}$  and  $\bar{\Delta}_{\bar{j}} = \frac{\bar{j}(\bar{j}+1)}{k+2}$ , respectively.

The complete system of local fields involved in the theory includes, besides the primary fields  $\Phi^{j\bar{j}}$ , all the fields of the form

$$J_{n_1}^{\alpha_1} \dots J_{n_N}^{\alpha_N} \bar{J}_{\bar{n}_1}^{\beta_1} \dots \bar{J}_{\bar{n}_M}^{\beta_M} \Phi^{j\bar{j}} \quad , \quad (4)$$

where  $J_n^\alpha$ ,  $\bar{J}_{\bar{n}}^\beta$  are the Laurent series components of  $J^\alpha(z)$  and  $\bar{J}^\beta(\bar{z})$ , respectively. Following [1], I shall denote by  $[\Phi^{j\bar{j}}]$  the hole set of the fields (4) associated with the primary field  $\Phi^{j\bar{j}}$ . From mathematical point of view  $[\Phi^{j\bar{j}}]$  corresponds to the highest weight representation of  $\hat{sl}_2 \times \hat{sl}_2$ .

In this work I will consider only the diagonal embedding the physical space of states into a tensor product of two chiral space of states. Such models are known as "A" series. Since for these models all primary fields are spinless, i.e.  $\bar{j} \equiv j(\bar{\Delta} \equiv \Delta)$ , I suppress  $\bar{j}$ -dependence below.

In [5] Kac and Kazhdan found that the highest weight representation of  $\hat{sl}_2$  is reducible if the highest weight  $j$  takes the values  $j_{n.m}$  defined by

$$j_{n.m}^+ = \frac{n-1}{2}j_- + \frac{m-1}{2}j_+ \quad \text{or} \quad j_{n.m}^- = -\frac{n}{2}j_- - \frac{m+1}{2}j_+ \quad , \quad (5)$$

with  $j_+ = 1$ ,  $j_- = -k - 2$ ,  $k \in \mathbb{C}$ ,  $\{n, m\} \in \mathbb{N}$ .

In general, given representations of a chiral algebra (symmetry algebra), to define fields of conformal field theory, one needs a construction attaching representation to a point. In [6] Feigin and Malikov proposed the improved construction for the weights given by (5). The point is that one more parameter should be introduced and a module should be attached to a pair. The first parameter is a point on a curve. As to the second, it can be taken as an isotopic coordinate <sup>1</sup>. Actually, this has a very simple physical interpretation. Since  $\Delta$  are quadratic in  $j$  one has to introduce additional parameter in order to define OP expansions (12) unambiguously otherwise they are defined up to  $j = -j - 1$  identification. Note that for the integrable representations the OP algebra is closed for  $0 \leq j \leq k/2$  with  $k \in \mathbb{N}$  so the OP expansions are well defined without  $x$ . However this is not the case for a general  $j$  defined by (5).

Let me now define  $S_j^\alpha$  as

$$S_j^- = \frac{\partial}{\partial x}, \quad S_j^0 = -x \frac{\partial}{\partial x} + j, \quad S_j^+ = -x^2 \frac{\partial}{\partial x} + 2jx, \quad (6)$$

The same definitions with substitutions  $x \rightarrow \bar{x}$ ,  $S_j^\alpha \rightarrow \bar{S}_j^\alpha$  are also valid. In above the isotopic coordinates  $x, \bar{x}$  were introduced. Together with  $z, \bar{z}$  they form the Malikov-Feigin pair.

The chiral currents (1) can be turned into a form (current)

$$J(x, z) = J^+(z) - 2xJ^0(z) - x^2J^-(z) \quad . \quad (7)$$

It is easily shown that the OP expansion of  $J(x, z)$  is

$$J(x_1, z_1)J(x_2, z_2) = -k \frac{x_{12}^2}{z_{12}^2} - 2 \frac{x_{12}}{z_{12}} J(x_2, z_2) - \frac{x_{12}^2}{z_{12}} \frac{\partial}{\partial x_2} J(x_2, z_2) + O(1) \quad , \quad (8)$$

where  $z_{ij} = z_i - z_j$ . The same OP expansion, of course, is valid for antiholomorphic current.

Define the primary fields as

$$J(x_1, z_1)\Phi^j(x_2, \bar{x}_2, z_2, \bar{z}_2) = -2j \frac{x_{12}}{z_{12}} \Phi^j(x_2, \bar{x}_2, z_2, \bar{z}_2) - \frac{x_{12}^2}{z_{12}} \frac{\partial}{\partial x_2} \Phi^j(x_2, \bar{x}_2, z_2, \bar{z}_2) + O(1) \quad . \quad (9)$$

It should be noted that in general case the primary fields are non-polynomial in  $x, \bar{x}$ . Furthermore,  $J(x, z)$  is not primary.

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<sup>1</sup>It should be noted that the primary fields including dependence on the isotopic coordinate have also been considered by Furlan et al.[7].

The highest weight representations of  $\hat{sl}_2 \times \hat{sl}_2$  are built as in (4) with substitutions  $J_n^\alpha \rightarrow J_n^\alpha(x)$ ,  $\bar{J}_n^\alpha \rightarrow \bar{J}_n^\alpha(\bar{x})$  and  $\Phi^{j\bar{j}} \rightarrow \Phi^j$ . The  $x(\bar{x})$ -dependent components of  $J(x, z)(\bar{J}(\bar{x}, \bar{z}))$  are given by

$$J_n^-(x) = J_n^- \quad , \quad J_n^0(x) = J_n^0 + xJ_n^- \quad , \quad J_n^+(x) = J_n^+ - 2xJ_n^0 - x^2J_n^- \quad . \quad (10)$$

It is evident that  $J^\alpha(x)$  form the Kac-Moody algebra

$$[J_n^\alpha(x), J_m^\beta(x)] = f_\gamma^{\alpha\beta} J_{n+m}^\gamma(x) + \frac{k}{2} nq^{\alpha\beta} \delta_{n+m} \quad .$$

The Operator Product of any two operators is given by

$$\phi^{j_1}(x, \bar{x}, z, \bar{z}) \phi^{j_2}(0, 0, 0, 0) = \sum_{j_3} C_{j_3}^{j_1 j_2}(x, \bar{x}, z, \bar{z}) \phi^{j_3}(0, 0, 0, 0) \quad . \quad (11)$$

It is well-known that all the coefficient functions  $C_{j_3}^{j_1 j_2}(x, \bar{x}, z, \bar{z})$  in the expansion (11) can be expressed via the weights(conformal dimensions) of the primary fields(basic operators) and the structure constants of Operator Algebra [1, 3]. The structure constants are defined as coefficients at the primary fields in the OP expansion<sup>2</sup>

$$\Phi^{j_1}(x, \bar{x}, z, \bar{z}) \Phi^{j_2}(0, 0, 0, 0) = \sum_{j_3} \frac{|x|^{2(j_1+j_2-j_3)}}{|z|^{2(\Delta_{j_1}+\Delta_{j_2}-\Delta_{j_3})}} C_{j_3}^{j_1 j_2} \Phi^{j_3}(0, 0, 0, 0) \quad . \quad (12)$$

The normalized two and three point functions of the primary fields can be represented as

$$\begin{aligned} \langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \Phi^{j_2}(x_2, \bar{x}_2, z_2, \bar{z}_2) \rangle &= \delta^{j_1 j_2} \frac{|x_{12}|^{4j_1}}{|z_{12}|^{4\Delta_{j_1}}} \quad , \\ \langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \Phi^{j_2}(x_2, \bar{x}_2, z_2, \bar{z}_2) \Phi^{j_3}(x_3, \bar{x}_3, z_3, \bar{z}_3) \rangle &= C^{j_1 j_2 j_3} \prod_{n < m} \frac{|x_{nm}|^{2\gamma_{nm}(j)}}{|z_{nm}|^{2\gamma_{nm}(\Delta)}} \quad , \end{aligned} \quad (13)$$

where  $y_{nm} = y_n - y_m$ ,  $\gamma_{12}(y) = y_1 + y_2 - y_3$ ,  $\gamma_{13}(y) = y_1 + y_3 - y_2$ ,  $\gamma_{23}(y) = y_2 + y_3 - y_1$ .

As to four point function, one can find it in the following form(see [3])

$$\langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \dots \Phi^{j_4}(x_4, \bar{x}_4, z_4, \bar{z}_4) \rangle = G^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z}) \prod_{n < m} \frac{|x_{nm}|^{2\gamma_{nm}(j)}}{|z_{nm}|^{2\gamma_{nm}(\Delta)}} \quad , \quad (14)$$

with  $\gamma_{14}(y) = 2y_1$ ,  $\gamma_{23}(y) = y_1 + y_2 + y_3 - y_4$ ,  $\gamma_{24}(y) = -y_1 + y_2 - y_3 + y_4$ ,  $\gamma_{34}(y) = -y_1 - y_2 + y_3 + y_4$  and

$$x = \frac{x_{12}x_{34}}{x_{14}x_{32}}, \quad \bar{x} = \frac{\bar{x}_{12}\bar{x}_{34}}{\bar{x}_{14}\bar{x}_{32}}, \quad z = \frac{z_{12}z_{34}}{z_{14}z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}} \quad .$$

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<sup>2</sup>For simplicity I set  $j \in \mathbb{R}$ . However, the generalization to  $j \in \mathbb{C}$  is straightforward.

In order to write down (14) explicitly one needs to calculate  $G^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$ . To this purpose I use the remarkable relation between the KZ equation [8] with the generators of  $sl_2$  determined by (6) and the differential equations of degenerate conformal field theories [1]. Due to this the four point function (14) is expressed via a five point function of the DF model, where  $x, \bar{x}$  play a role of the coordinate of the fifth operator. For further details I refer to the original work [3]. Although the relation was discovered by Fateev and Zamolodchikov for the  $SU(2)$  WZW models and the minimal models it proves that the same result is valid in the case of the degenerate  $SL(2)$  conformal field theories and the Dotsenko-Fateev models ( $\alpha_-^2 \in \mathbb{C}$ ) [9].

Let  $\Phi^j(x, \bar{x}, z, \bar{z}) = \Phi_+^j(x, \bar{x}, z, \bar{z})$  with  $j = j^+$  and  $\Phi^j(x, \bar{x}, z, \bar{z}) = \Phi_-^j(x, \bar{x}, z, \bar{z})$  with  $j = j^-$ , where  $j^\alpha$  are defined in (5). The functions  $G^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$  are given by

$$\begin{aligned} G^{(A)}(x, \bar{x}, z, \bar{z}) &= Z^{(A)}(j_1, j_2, j_3, j_4) |z|^a |1-z|^b \prod_{i=1}^{n_1-1} \prod_{i'=1}^{m_1-1} \int d^2 u_i \int d^2 w_{i'} |u_i - w_{i'}|^{-4} \times \\ &\times \prod_{i=1}^{n_1-1} |u_i|^{4\alpha_2^{(A)}\alpha_-} |1-u_i|^{4\alpha_3^{(A)}\alpha_-} |x-u_i|^{4\alpha_{21}\alpha_-} |z-u_i|^{4\alpha_1^{(A)}\alpha_-} \prod_{i < i'}^{n_1-1} |u_{ii'}|^{4\alpha_-^2} \times \\ &\times \prod_{i=1}^{m_1-1} |w_i|^{4\alpha_2^{(A)}\alpha_+} |1-w_i|^{4\alpha_3^{(A)}\alpha_+} |x-w_i|^{4\alpha_{21}\alpha_+} |z-w_i|^{4\alpha_1^{(A)}\alpha_+} \prod_{i < i'}^{m_1-1} |w_{ii'}|^{4\alpha_+^2}. \end{aligned} \quad (15)$$

Here  $G^{(1)}(x, \bar{x}, z, \bar{z}) = G_{++++}^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$ ,  $G^{(2)}(x, \bar{x}, z, \bar{z}) = G_{+++-}^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$ ,  $G^{(3)}(x, \bar{x}, z, \bar{z}) = G_{++--}^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$ ,  $G^{(4)}(x, \bar{x}, z, \bar{z}) = G_{+---}^{j_1, j_2, j_3, j_4}(x, \bar{x}, z, \bar{z})$ ,  $a = 4j_1 j_2 \alpha_+^2$ ,  $b = 4j_1 j_3 \alpha_+^2$ ,  $\alpha_- = -\sqrt{k+2}$ ,  $\alpha_+ \alpha_- = -1$ . Furthermore, the  $\alpha_i^{(A)}$ 's are defined via  $\alpha_i^{(A)} = \frac{1-N_i^{(A)}}{2}\alpha_- + \frac{1-M_i^{(A)}}{2}\alpha_+$  with

$$\begin{aligned} N_1^{(1)} &= N_2^{(3)} = \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} + \frac{n_4}{2} - 1, & M_1^{(1)} &= M_2^{(3)} = \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} + \frac{m_4}{2}; \\ N_2^{(1)} &= N_1^{(3)} = \frac{n_1}{2} + \frac{n_2}{2} - \frac{n_3}{2} - \frac{n_4}{2}, & M_2^{(1)} &= M_1^{(3)} = \frac{m_1}{2} + \frac{m_2}{2} - \frac{m_3}{2} - \frac{m_4}{2}; \\ N_3^{(1)} &= -N_4^{(3)} = \frac{n_1}{2} - \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2}, & M_3^{(1)} &= -M_4^{(3)} = \frac{m_1}{2} - \frac{m_2}{2} + \frac{m_3}{2} - \frac{m_4}{2}; \\ N_4^{(1)} &= -N_3^{(3)} = -\frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2}, & M_4^{(1)} &= -M_3^{(3)} = -\frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} - \frac{m_4}{2}; \end{aligned}$$

$$\begin{aligned}
N_1^{(2)} = -N_4^{(4)} &= \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} - \frac{1}{2} \quad , \quad M_1^{(2)} = -M_4^{(4)} = \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} - \frac{m_4}{2} \quad ; \\
N_2^{(2)} = N_3^{(4)} &= \frac{n_1}{2} + \frac{n_2}{2} - \frac{n_3}{2} + \frac{n_4}{2} - \frac{1}{2} \quad , \quad M_2^{(2)} = M_3^{(4)} = \frac{m_1}{2} + \frac{m_2}{2} - \frac{m_3}{2} + \frac{m_4}{2} \quad ; \\
N_3^{(2)} = N_2^{(4)} &= \frac{n_1}{2} - \frac{n_2}{2} + \frac{n_3}{2} + \frac{n_4}{2} - \frac{1}{2} \quad , \quad M_3^{(2)} = M_2^{(4)} = \frac{m_1}{2} - \frac{m_2}{2} + \frac{m_3}{2} + \frac{m_4}{2} \quad ; \\
N_4^{(2)} = -N_1^{(4)} &= -\frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} + \frac{n_4}{2} - \frac{1}{2} \quad , \quad M_4^{(2)} = -M_1^{(4)} = -\frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} + \frac{m_4}{2} .
\end{aligned}$$

In order to take into account a relative normalization between the operators of the DF models and the ones of the  $SL(2)$  degenerate conformal field theories one has to introduce the normalization constants  $Z^{(A)}(j_1, j_2, j_3, j_4)$ . Up to irrelevant factors they may be written as follows

$$Z^{(1)}(j_1, j_2, j_3, j_4) = V(|N_1^{(1)}|, |M_1^{(1)}|) \prod_{\{2,3,4\}} V(|N_i^{(1)}| + 1, |M_i^{(1)}| + 1) \quad , \quad (16)$$

$$Z^{(2)}(j_1, j_2, j_3, j_4) = V(|N_4^{(2)}| + 1, |M_4^{(2)}| + 1) \prod_{\{1,2,3\}} V(|N_i^{(2)}|, |M_i^{(2)}|) \quad , \quad (17)$$

$$Z^{(3)}(j_1, j_2, j_3, j_4) = V(|N_2^{(3)}|, |M_2^{(3)}|) \prod_{\{1,3,4\}} V(|N_i^{(3)}| + 1, |M_i^{(3)}| + 1) \quad , \quad (18)$$

$$Z^{(4)}(j_1, j_2, j_3, j_4) = V(|N_1^{(4)}| + 1, |M_1^{(4)}| + 1) \prod_{\{2,3,4\}} V(|N_i^{(4)}|, |M_i^{(4)}|) \quad , \quad (19)$$

where

$$V(n, m) = \rho^{-(n-1)(m-1)} P(n, m) \quad , \quad \rho = \alpha_+^2 \quad , \quad \rho' = \alpha_-^2 \quad .$$

The function  $P(n, m)$  is given by

$$P(n, m) = \prod_{i=1}^{n-1} \prod_{j=1}^{m-1} [i\rho' - j]^{-2} \prod_{i=1}^{n-1} \frac{\Gamma[i\rho']}{\Gamma[1 - i\rho']} \prod_{j=1}^{m-1} \frac{\Gamma[j\rho]}{\Gamma[1 - j\rho]} \quad , \quad P(1, 1) = 1 \quad .$$

The 2D multiple integrals (15) are defined via contour integrals. This can be done by using the results of Dotsenko and Fateev [2, 10].

The structure constants are found in the form

$$\begin{aligned}
C_{+++}^{j_1, j_2, j_3} &= \frac{Z^{(1)}(0, j_1, j_2, j_3) \left( Z^{(1)}(0, 0, 0, 0) \right)^{\frac{1}{2}}}{\left( Z^{(1)}(0, j_1, j_1, 0) Z^{(1)}(0, j_3, 0, j_3) Z^{(1)}(0, 0, j_2, j_2) \right)^{\frac{1}{2}}} = \\
&= \left( \frac{\Gamma[\rho]}{\Gamma[1-\rho]} \right)^{\frac{1}{2}} P\left(\sigma' - \frac{1}{2}, \sigma + \frac{1}{2}\right) \times \\
&\times \prod_{\{1,2,3\}} (-)^{\frac{n_i-1}{2}} \rho^{(1-n_i)} \left( \frac{\Gamma[n_i - m_i \rho]}{\Gamma[1 - n_i + m_i \rho]} \right)^{\frac{1}{2}} \frac{P\left(\sigma' - n_i + \frac{1}{2}, \sigma - m_i + \frac{1}{2}\right)}{P(n_i, m_i)} \quad , \quad (20)
\end{aligned}$$

$$\begin{aligned}
C_{+-+}^{j_1, j_2, j_3} &\propto Z^{(2)}(0, j_1, j_2, j_3) = \\
&= \rho^{-\frac{1}{2}} P\left(\sigma', \sigma + \frac{1}{2}\right) \prod_{\{1,2,3\}} \rho^{-(n_i-1)(m_i-\frac{1}{2})} P\left(\sigma' - n_i, \sigma - m_i + \frac{1}{2}\right) \quad , \quad (21)
\end{aligned}$$

$$\begin{aligned}
C_{+--}^{j_1, j_2, j_3} &\propto \frac{Z^{(3)}(0, j_1, j_2, j_3)}{\left( Z^{(3)}(0, 0, j_2, j_2) Z^{(3)}(0, 0, j_3, j_3) \right)^{\frac{1}{2}}} = \\
&= \rho^{-(n_1-1)(m_1-\frac{1}{2})} \left( P(n_1, m_1 + 1) P(n_1, m_1) \right)^{\frac{1}{2}} P\left(\sigma' - \frac{1}{2}, \sigma + \frac{1}{2}\right) \times \\
&\times \prod_{\{1,2,3\}} \frac{P\left(\sigma' - n_i + \frac{1}{2}, \sigma - m_i + \frac{1}{2}\right)}{\left( P(n_i, m_i + 1) P(n_i, m_i) \right)^{\frac{1}{2}}} \quad , \quad (22)
\end{aligned}$$

$$\begin{aligned}
C_{---}^{j_1, j_2, j_3} &\propto Z^{(4)}(0, j_1, j_2, j_3) = \\
&= \rho^{-\frac{1}{2}} P\left(\sigma', \sigma + \frac{1}{2}\right) \prod_{\{1,2,3\}} \rho^{-(n_i-1)(m_i-\frac{1}{2})} P\left(\sigma' - n_i, \sigma - m_i + \frac{1}{2}\right) \quad . \quad (23)
\end{aligned}$$

Here  $\sigma' = \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2}$  and  $\sigma = \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2}$ . It should be noted that the normalization of (21-23), as defined in (13), is ambiguous because  $\Phi_-$  operators don't contain the unity operator. It belongs to the  $\Phi_+$  operators, namely  $\mathbf{1} = \Phi_+^{\mathbf{j}=0}$ .

Setting  $x = z, \bar{x} = \bar{z}$  in (15), which corresponds to the quantum hamiltonian reduction [7], one immediately obtain the relation between the proper 4-point functions of the DF model(see (28)). Due to  $Z_2$  symmetry of the model it is sufficient to consider the  $\Phi_+$

fields only. Using (20) I get the structure constants in the following symmetric form

$$C(n_1, m_1; n_2, m_2; n_3, m_3) = P^{\frac{1}{2}}(2, 2) P(\sigma' + \frac{1}{2}, \sigma + \frac{1}{2}) \times \\ \times \prod_{\{1,2,3\}} \frac{P(\sigma' - n_i + \frac{1}{2}, \sigma - m_i + \frac{1}{2})}{\left( P(n_i + 1, m_i + 1) P(n_i, m_i) \right)^{\frac{1}{2}}} . \quad (24)$$

From the set (5) it is worth to distinguish the so-called admissible representations [11], which correspond to the rational level  $k$ . In the case  $k = -2 + p/q$ , with the coprime integers  $p$  and  $q$ , it is possible to recover the minimal models (series with  $c < 1$ ) via the Drinfeld-Sokolov reduction. On the other hand  $k = -2 - p/q$  leads to the Liouville series with  $c > 25$ . The second point is an existence of modular invariants for such representations.

At the rational level  $k = -2 + p/q$  there is a symmetry  $j_{n,m}^- = j_{q-n+1, p-m}^+$  which allows one to reduce the  $\Phi_-$  fields to the  $\Phi_+$  ones. Up to the normalization factors one can identify the various  $C$ 's, namely  $C_{+++}^{j_1, j_2, j_3} = C_{+-+}^{j_1, j_2, j_3}$  and  $C_{++-}^{j_1, j_2, j_3} = C_{---}^{j_1, j_2, j_3}$ . This results in the following structure constants

$$C_{+++}^{j_1, j_2, j_3} = C_{+++}^{j_1, j_2, j_3} , \\ \text{and} \\ C_{+++}^{j_1, j_2, j_3} \propto C_{---}^{j_1, j_2, j_3} \quad \text{with} \quad n_i \rightarrow q - n_i + 1, \quad m_i \rightarrow p - m_i . \quad (25)$$

It is easy to see from (25) that the OP algebra at the rational level is closed in the grid  $1 \leq n_i \leq q, 1 \leq m_i \leq p - 1$ . The corresponding fusion rules are given by

$$\begin{cases} \max(n_{12} + 1, n_{21} + 1) \leq n_3 \leq \min(n_1 + n_2 - 1, 2q - n_1 - n_2 + 1) , \\ \max(m_{12} + 1, m_{21} + 1) \leq m_3 \leq \min(m_1 + m_2 - 1, 2p - m_1 - m_2 - 1) , \end{cases} \quad (26)$$

$$\begin{cases} \max(n_1 + n_2 - q + 1, q + 3 - n_1 - n_2) \leq n_3 \leq \min(n_{12} + q - 1, n_{21} + q - 1) , \\ \max(m_1 + m_2 - p + 1, p + 1 - m_1 - m_2) \leq m_3 \leq \min(m_{12} + p - 1, m_{21} + p - 1) , \end{cases} \quad (27)$$

As a checking procedure one can directly analyze singularities of (15). These fusion rules agree with those found in [12] from the differential equations for the conformal blocks. Later the same result was obtained by cohomological methods [6].

It should be also noted that in the case of  $n_i = 1, \rho = \frac{1}{k+2}, k \in \mathbb{N}$  and  $m_i = 2j_i + 1$  the structure constants and fusion rules of the unitary representations are recovered [3].

Let me now further investigate the theory at the rational level. In order to consider the level  $k$  defined by  $k = -2 - p/q$  one can proceed in complete accordance with the



previous case. A simple analysis shows the same fusion rules as (26-27). As to the structure constants, they look like (25). The only difference is the sign of  $\rho(\rho')$ .

Let me now conclude by giving some remarks.

(i) One is on the normalizations of the structure constants (21-23). Note that the usual normalization (13) is useless for this purpose, therefore it is necessary to look for something more. One can try to solve the bootstrap equations [1]. It could clarify the problem of picking up the right normalization, but this is beyond the scope of this note.

(ii) Another remark is on the Wakimoto free field representation for these models. The first attempts were made to extend the Wakimoto representation to the rational weights in [13, 14]. However all of them used the ordinary construction attaching representation to a point, i.e. without  $x$ . As a result their OP expansions were ambiguous, so the fusion rules are determined up to  $j = -j - 1$  identification. Recently, Petersen, Rasmussen and Yu developed the Wakimoto representation that relies upon introducing the isotopic coordinates related to the  $sl_2$  representations [15]. Using that construction, they found the conformal blocks on the sphere and the fusion rules like (26-27). The next step would be to build the correlation functions and to derive the OP algebra of the primary fields. The theory is then fully solved by the Wakimoto representation. Unfortunately the last step have not realized yet.

(iii) Impressive seems the following relation between the correlations functions of the DF model

$$\langle \Phi_{n_1.m_1} \Phi_{n_2.m_2} \Phi_{n_3.m_3} \Phi_{n_4.m_4} \rangle \propto \langle \Phi_{N_1.M_1} \Phi_{N_2.M_2} \Phi_{N_3.M_3} \Phi_{N_4.M_4} \rangle \quad , \quad (28)$$

where

$$\begin{aligned} N_1^{(1)} &= \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} + \frac{n_4}{2} \quad , & M_1^{(1)} &= \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} + \frac{m_4}{2} \quad ; \\ N_2^{(1)} &= \frac{n_1}{2} + \frac{n_2}{2} - \frac{n_3}{2} - \frac{n_4}{2} \quad , & M_2^{(1)} &= \frac{m_1}{2} + \frac{m_2}{2} - \frac{m_3}{2} - \frac{m_4}{2} \quad ; \\ N_3^{(1)} &= \frac{n_1}{2} - \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} \quad , & M_3^{(1)} &= \frac{m_1}{2} - \frac{m_2}{2} + \frac{m_3}{2} - \frac{m_4}{2} \quad ; \\ N_4^{(1)} &= -\frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} \quad , & M_4^{(1)} &= -\frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} - \frac{m_4}{2} \quad . \end{aligned}$$

The problem is to understand what underlies this mysterious relation. May be there is a hidden symmetry in the theory.

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